

# On the distance preserving trees in graphs

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## Abstract

For a vertex  $v$  of a graph  $G$ , a spanning tree  $T$  of  $G$  is distance-preserving from  $v$  if, for any vertex  $w$ , the distance from  $v$  to  $w$  on  $T$  is the same as the distance from  $v$  to  $w$  on  $G$ . If two vertices  $u$  and  $v$  are distinct, then two distance-preserving spanning trees  $T_u$  from  $u$  and  $T_v$  from  $v$  are distinct in general. A purpose of this paper is to give a characterization for a given weighted graph  $G$  to have a spanning tree  $T$  such that  $T$  is a distance-preserving spanning tree from distinct two vertices.

## 1 Introduction

Let  $G$  be a simple undirected graph. The vertex set and the edge set of  $G$  is denoted by  $V(G)$  and  $E(G)$ , respectively. For a subset  $U \subseteq V(G)$ , the subgraph induced by  $U$  is denoted by  $G[U]$ . A *weighted graph* is a graph each edge of whose edges is assigned a real number (called the *cost* or *weight* of the edge). We denote the weight of an edge  $e$  of  $G$  by  $w(e)$ . For a path  $P$  of  $G$ , the *length* of  $P$  is defined as the sum of the weights of its edges. The *distance* between two vertices  $u$  and  $v$  of a graph  $G$  is the minimum length of paths from  $u$  to  $v$ , and is denoted by  $d_G(u, v)$ . For a subset of vertices  $S$ , the distance from  $u$  to  $S$  is defined by

$$d_G(u, S) = \min_{v \in S} d_G(u, v).$$

Let  $v$  be a vertex of  $G$ . A spanning tree  $T$  of  $G$  is a *distance-preserving spanning tree* (or a *DP-tree* for short) from  $v$  if  $d_T(v, w) = d_G(v, w)$  for each  $w \in V(G)$ . An example of a DP-tree  $T$  from  $u$  in a graph  $G$  is shown in Fig. 1.

In a well-known book “Graphs and Digraphs” written by Chartrand, Lesniak, and Zhang [1], we can find an exercise of Section 2.3: “Give an example of a connected graph  $G$  that is not a tree and two vertices  $u$  and  $v$  of  $G$  such that a distance-preserving spanning tree from  $v$  is the same as a distance-preserving spanning tree from  $u$ .” In Fig. 1, the spanning tree  $T$  is distance-preserving from the two vertices  $u$  and  $v$ . Hence Fig. 1 is an answer the question.

A purpose of this paper is to give a complete answer of the question. That is, for a given weighted graph  $G$  and two vertices  $u$  and  $v$ , we would like to give a characterization for a graph  $G$  to have a spanning tree  $T$  such that  $T$  is a distance-preserving spanning tree from  $u$  as well as from  $v$ .

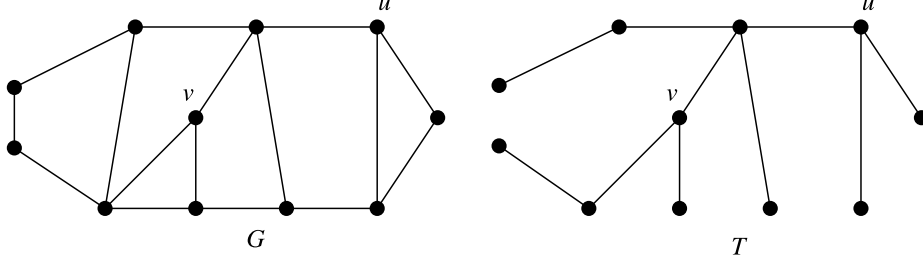


Figure 1: An example of a distance-preserving tree  $T$  from  $u$  of a graph  $G$ . We assume that the weights of edges are 1. The tree  $T$  is also a distance-preserving tree from  $v$ .

## 2 Main result

In this section, we show the following theorem. If a spanning tree  $T$  of  $G$  is a distance-preserving spanning tree from  $u$  as well as from  $v$ , we say that  $T$  is a *common distance-preserving spanning tree* of  $u$  and  $v$  in  $G$ .

**Theorem 2.1.** *Let  $G$  be a weighted graph and  $u$  and  $v$  are two vertices of  $G$ . A spanning tree  $T$  of  $G$  is a common distance-preserving spanning tree of  $u$  and  $v$  if and only if the following three conditions hold.*

- (1) *A shortest  $u$ - $v$  path  $P$  in  $G$  is unique.*
- (2) *We define the unique shortest  $u$ - $v$  path as  $P = (u = v_0, v_1, \dots, v_k = v)$ . For any vertex  $x$ , there is a unique vertex  $v_i \in V(P)$  such that  $d_G(x, v_i) = d_G(x, V(P))$ .*
- (3) *For  $0 \leq i \leq k$ , let  $V_i = \{x \mid x \in V(G) \text{ and } d_G(x, v_i) = d_G(x, V(P))\}$ . If  $e = xy \in E(G)$  for  $x \in V_i$  and  $y \in V_j$ , then  $w(e) \geq d_G(v_i, v_j)$  and  $|d_G(v_i, x) - d_G(v_j, y)| \leq w(e) - d_G(v_i, v_j)$ .*

We first show the necessary condition of Theorem 2.1.

**Lemma 2.2.** *If  $u$  and  $v$  have a common distance-preserving spanning tree  $T$  in  $G$ , a shortest  $u$ - $v$  path is unique.*

*Proof.* Let  $P$  be the  $u$ - $v$  path in  $T$ . Since  $T$  is distance-preserving from  $v$ ,  $P$  is a shortest  $u$ - $v$  path. Assume to the contrary that there is another shortest  $u$ - $v$  path  $P_1$ . Then there is a vertex  $x$  on  $P_1$  but not on  $P$ .

Let  $P_x = (u = x_1, x_2, \dots, x_k = x)$  be the  $u$ - $x$  path in  $T$ . Let  $x_i$  be the vertex of  $P_x$  such that  $x_{i-1}$  is on  $P$  but  $x_i$  is not on  $P$ . Since  $P_x$  is a shortest  $u$ - $x$  path, we have  $d_T(u, x) = d_T(u, x_i) + d_T(x_i, x)$ . Similarly, we obtain  $d_T(x, v) =$

$d_T(x, x_i) + d_T(x_i, v)$ . Since  $x$  is a vertex on the shortest  $u$ - $v$  path, we have

$$\begin{aligned}
d_G(u, v) &= d_G(u, x) + d_G(x, v) \\
&= d_T(u, x) + d_T(x, v) \\
&= d_T(u, x_i) + d_T(x_i, x) + d_T(x, x_i) + d_T(x_i, v) \\
&= d_T(u, v) + 2d_T(x, x_i) \\
&= d_G(u, v) + 2d_T(x, x_i).
\end{aligned}$$

Thus  $d_T(x, x_i) = 0$ , and hence a contradiction is obtained.  $\square$

If  $u$  and  $v$  have a common distance-preserving spanning tree  $T$ , by Lemma 2.2, there is a unique shortest  $u$ - $v$  path  $P = (u = v_0, v_1, \dots, v_k = v)$ . By the proof of Lemma 2.2, the unique  $u$ - $v$  path in  $T$  is the unique shortest  $u$ - $v$  path  $P$  in  $G$ .

**Lemma 2.3.** *Assume that  $u$  and  $v$  have a common distance-preserving spanning tree  $T$  in  $G$ . Let  $P = (u = v_0, v_1, \dots, v_k = v)$  be the unique shortest  $u$ - $v$  path in  $G$ . For any vertex  $x$  of  $G$ , there is a unique vertex  $v_i$  of  $P$  such that  $d_G(x, v_i) = d_G(x, V(P))$ .*

*Proof.* Let  $x$  be a vertex of  $G$ . If  $x$  is on  $P$ , the lemma is trivially true. So we assume that  $x \notin V(P)$ . Let  $P_x$  be the  $u$ - $x$  path of  $T$ . Since  $T$  is distance-preserving from  $u$ ,  $P_x$  is a shortest path from  $u$  to  $x$ . Hence  $d_G(u, x) = d_G(u, w) + d_G(w, x)$  for every vertex  $w$  of  $P_x$ .

Since  $u = v_0 \in V(P)$  and  $x \notin V(P)$ , the path  $P_x$  contains a unique vertex  $v_i \in V(P)$  such that  $v_l \notin V(P)$  for every  $l > i$  (if  $P_x$  has  $v_k = v$ , then  $v_i = k$ ). For  $0 \leq j \leq i$ , we have  $d_T(u, v_0) < d_T(u, v_1) < \dots < d_T(u, v_i)$ . Since  $T$  is distance-preserving from  $u$ , we have

$$d_G(u, v_0) < d_G(u, v_1) < \dots < d_G(u, v_i). \quad (1)$$

Since  $P_x$  is a shortest  $u$ - $x$  path in  $G$ , for  $0 \leq j \leq i$ , we have  $d_G(u, x) = d_G(u, v_j) + d_G(v_j, x)$ . Thus  $d_G(x, v_j) = d_G(u, x) - d_G(u, v_j)$  for  $0 \leq j \leq i$ . Therefore, by (1), we obtain

$$d_G(x, v_i) < d_G(x, v_{i-1}) < \dots < d_G(x, v_0) = d_G(x, u).$$

Similarly, since  $T$  is distance-preserving from  $v$ , for  $i \leq l \leq k$ , we obtain  $d_G(x, v_l) = d_G(v, x) - d_G(v_l, v)$ , and thus we obtain

$$d_G(x, v_i) < d_G(x, v_{i+1}) < \dots < d_G(x, v_k) = d_G(x, v).$$

Hence the vertex  $v_i$  is the unique nearest vertex in  $P$  from  $x$ , we obtain  $d_G(x, v_i) = d_G(x, V(P))$ .  $\square$

For  $0 \leq i \leq k$ , we define

$$V_i = \{x \mid x \in V(G) \text{ and } d_G(x, v_i) = d_G(x, V(P))\}, \quad (2)$$

where  $P = (u = v_0, v_1, \dots, v_k = v)$  is the unique shortest  $u$ - $v$  path defined in Lemma 2.3.

By Lemma 2.3, we can see that  $V_i \cap V_j = \emptyset$  and  $V_0 \cup V_1 \cup \dots \cup V_k = V(G)$ . That is,  $V_0 \cup V_1 \cup \dots \cup V_k$  is a partition of  $V(G)$ .

By the proof of Lemma 2.3, if  $x \in V_i$ , the  $u$ - $x$  path  $P_x$  in  $T$  contains  $v_i$ , and  $P_x$  is also a shortest  $u$ - $x$  path of  $G$ . Hence, for every  $x \in V_i$ , we have  $d_G(v_i, x) = d_T(v_i, x)$ .

**Lemma 2.4.** *Let  $V_0 \cup V_1 \cup \dots \cup V_k$  be a partition defined by (2). If  $e = xy \in E(G)$  for  $x \in V_i$  and  $y \in V_j$ , then  $w(e) \geq d_G(v_i, v_j)$  and  $|d_G(v_i, x) - d_G(v_j, y)| \leq w(e) - d_G(v_i, v_j)$ .*

*Proof.* If  $i = j$ , the lemma is true. Hence we assume that  $i < j$ .

Since  $x \in V_i$  is adjacent to  $y \in V_j$ , we obtain

$$d_G(u, y) \leq d_G(u, x) + w(e). \quad (3)$$

Since  $T$  is distance-preserving from  $u$ , we have

$$\begin{aligned} d_T(u, x) &= d_T(u, v_i) + d_T(v_i, x), \\ d_T(u, y) &= d_T(u, v_j) + d_T(v_j, y). \end{aligned}$$

Thus, by (3),  $d_T(u, v_j) + d_T(v_j, y) \leq d_T(u, v_i) + d_T(v_i, x) + w(e)$ . Since  $d_T(u, v_j) - d_T(u, v_i) = d_T(v_i, v_j) = d_G(v_i, v_j)$ , we obtain

$$d_G(v_i, v_j) \leq d_T(v_i, x) - d_T(v_j, y) + w(e). \quad (4)$$

Similarly, by considering the fact that  $T$  is distance-preserving from  $v$ , we obtain

$$d_G(v_i, v_j) \leq d_T(v_j, y) - d_T(v_i, x) + w(e). \quad (5)$$

By adding the both side of inequalities (4) and (5), we have  $d_G(v_i, v_j) \leq w(e)$ .

From (4) and the fact  $d_G(v_i, x) = d_T(v_i, x)$ , we obtain

$$d_G(v_i, x) - d_G(v_j, y) \geq -(w(e) - d_G(v_i, v_j)).$$

Similarly, from (5),

$$d_T(v_i, x) - d_T(v_j, y) \leq w(e) - d_G(v_i, v_j).$$

Thus we obtain

$$|d_G(v_i, x) - d_G(v_j, y)| \leq w(e) - d_G(v_i, v_j).$$

□

By Lemmas 2.2, 2.3 and 2.4, we have shown the necessary condition in Theorem 2.1.

Next we prove the sufficiency of Theorem 2.1. We assume that two vertices  $u$  and  $v$  in  $G$  satisfy the following three conditions.

- (1) A shortest  $u$ - $v$  path  $P$  in  $G$  is unique.
- (2) We define the shortest  $u$ - $v$  path as  $P = (u = v_0, v_1, \dots, v_k = v)$ . For any vertex  $x$ , there is a unique vertex  $v_i \in V(P)$  such that  $d_G(x, v_i) = d_G(x, V(P))$ .
- (3) For  $0 \leq i \leq k$ , let  $V_i = \{x \mid x \in V(G) \text{ and } d_G(x, v_i) = d_G(x, V(P))\}$ . If  $e = xy \in E(G)$  for  $x \in V_i$  and  $y \in V_j$ , then  $w(e) \geq d_G(v_i, v_j)$  and  $|d_G(v_i, x) - d_G(v_j, y)| \leq w(e) - d_G(v_i, v_j)$ .

For  $0 \leq i \leq k$ , let  $G_i$  be the subgraph of  $G$  induced by  $V_i$ .

**Lemma 2.5.** *For  $0 \leq i \leq k$ , the induced subgraph  $G_i = G[V_i]$  is connected.*

*Proof.* Assume that  $G_i$  is disconnected for some  $i$ . Let  $x$  be a vertex in a component that does not contain  $v_i$ . So, a shortest  $x$ - $v_i$  path  $P_x$  of  $G$  have to contain an edge  $e = yw$  such that  $y \in V_i$  and  $w \in V_j$  for  $j \neq i$ . Since  $P_x$  is a shortest  $x$ - $v_i$  path  $G$ , we have  $d_G(x, v_i) = d_G(x, w) + d_G(w, v_i)$ . By the definition of  $V_i$ , we have  $d_G(w, v_j) < d_G(w, v_i)$ . Hence, we obtain

$$\begin{aligned} d_G(x, v_i) &= d_G(x, w) + d_G(w, v_i) \\ &> d_G(x, w) + d_G(w, v_j) \\ &\geq d_G(x, v_j). \end{aligned}$$

This contradicts the fact that  $x \in V_i$ . □

**Lemma 2.6.** *For  $0 \leq i \leq k$  and any vertex  $x \in V_i$ ,*

$$d_{G_i}(v_i, x) = d_G(v_i, x).$$

*Proof.* Since  $G_i$  is a connected subgraph of  $G$ , clearly  $d_{G_i}(v_i, x) \geq d_G(v_i, x)$ . Assume that there is a vertex  $x \in V_i$  such that  $d_{G_i}(v_i, x) > d_G(v_i, x)$ .

In this case, a shortest  $v_i$ - $x$  path contains a vertex  $y \in V_j$  and  $j \neq i$ . Hence

$$\begin{aligned} d_G(v_i, x) &= d_G(v_i, y) + d_G(y, x) \\ &> d_G(v_j, y) + d_G(y, x) \\ &\geq d_G(v_j, x). \end{aligned}$$

This contradicts the fact that  $x \in V_i$ . □

Now we are ready to prove the sufficiency of Theorem 2.1.

*Proof of Sufficiency.* By Lemma 2.5,  $G_i$  is connected. So,  $G_i$  has a distance-preserving spanning tree  $T_i$  from  $v_i$ . We define a spanning tree  $T$  of  $G$  that has the edge set

$$E(T) = E(P) \cup E(T_0) \cup E(T_1) \cup \dots \cup E(T_k), \quad (6)$$

where  $P$  is the unique shortest  $u$ - $v$  path of  $G$ . We can see easily that  $T$  is a spanning tree of  $G$ .

We show that the tree  $T$  is a common distance-preserving spanning tree of  $u$  and  $v$ . That is, for any vertex  $x$ , we show that  $d_T(u, x) = d_G(u, x)$  and  $d_T(v, x) = d_G(v, x)$ . In this proof, we show that  $T$  is distance-preserving from  $u$ . We can prove similarly  $T$  is distance-preserving from  $v$ .

For a vertex  $x$ , mappings  $p$  and  $h$  are defined as

$$\begin{cases} p(x) = d_G(u, v_i), & \text{when } x \in V_i, \\ h(x) = d_G(v_i, x), & \text{when } x \in V_i, \end{cases}$$

and then define  $W(x) = p(x) + h(x)$ . It is easy to see that  $d_T(u, x) = p(x) + h(x)$  for any  $x$ . By the definition,  $W(u_0) = 0 + 0 = 0$ . Let  $P_x = (u = u_0, u_1, \dots, u_s = x)$  be a shortest  $u$ - $x$  path in  $G$ . Since  $P$  is a shortest path, we have  $d_G(u, u_{i+1}) = d_G(u, u_i) + w(e_i)$ , where  $e_i = u_i u_{i+1}$ . Consider the sequence  $W(u_0), W(u_1), \dots, W(u_s)$  and the value of  $|W(u_{i+1}) - W(u_i)|$ .

We first assume that the edge  $e_i = u_i u_{i+1}$  is a edge of  $T$ . If  $e_i$  is in  $P$ ,  $|W(u_{i+1}) - W(u_i)| = |p(u_{i+1}) - p(u_i)| = w(e_i)$ . If  $e_i$  is  $T_i$ ,  $|W(u_{i+1}) - W(u_i)| = |h(u_{i+1}) - h(u_i)| = w(e_i)$ . Thus we have  $|W(u_{i+1}) - W(u_i)| = w(e_i)$  when  $e_i$  is in  $T$ .

Next we suppose that  $e_i$  is not in  $T$ . If  $u_i \in V_j$  and  $u_{i+1} \in V_{j'}$ , by the condition (3), we obtain

$$\begin{aligned} |W(u_{i+1}) - W(u_i)| &= |(p(u_{i+1}) - p(u_i)) + (h(u_{i+1}) - h(u_i))| \\ &= |d_G(v_{j'}, v_j) + (d_G(v_{j'}, u_{i+1}) - d_G(v_j, u_i))| \\ &\leq d_G(v_{j'}, v_j) + w(e_i) - d_G(v_{j'}, v_j) \text{ (by condition (3))} \\ &= w(e_i). \end{aligned}$$

In both cases, we obtain  $|W(u_{i+1}) - W(u_i)| \leq w(e_i)$ . Hence,

$$\begin{aligned} d_T(u, x) &= W(u_s) - W(u_0) \\ &= \sum_{0 \leq i \leq s-1} (W(u_{i+1}) - W(u_i)) \\ &\leq \sum_{0 \leq i \leq s-1} w(e_i) \\ &= d_G(u, x). \end{aligned}$$

Since  $T$  is a connected subgraph of  $G$ , we have  $d_T(u, x) \geq d_G(u, x)$ . Thus, we obtain  $d_T(u, x) = d_G(u, x)$  for any vertex  $x$ .  $\square$

We have completed the proof of Theorem 2.1.

## References

- [1] G. Chartrand, L. Lesniak, P. Zhang, Graphs & Digraphs, 5th ed., Chapman & Hall/CRC, 2011.